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THE USE OF QUATERNIONS TO GENERALIZE THE KOLOSOV-MUSKHELISHVILI METHOD TO THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY*

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A boundary equation is obtained for the first fundamental problem of the theory of elasticity using the Kolosov-Muskhelishvili method, by replacing the field of complex numbers by the field of Hamiltonian quaternions. A specific solution is given for the space with an elliptical cavity, subjected to a uniform tensile force at infinity.

The elements of the algebra of generalized quaternions /1/ are as follows: $\eta = ex_0 + xi + yj + zk$. In what follows, we shall regard ex_0 as time, i.e. $ex_0 = t$. Thus we can write any function $f(\eta)$ in the form of an analytic function

$$f(\eta) = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k,\ \alpha_n = \alpha_n(t,\ x,\ y,\ z),\ n = 1,\ \ldots,\ 4$$

where α_1 is the scalar part, and the sum of the remaining terms is the vector part.

We can further show that

$$\Box\alpha_n = 0,\ \Box = \partial_{tt} + \partial_{xx} + \partial_{yy} + \partial_{zz} \tag{1}$$

The solutions of Eq.(1) are harmonic functions. Using different manipulations, we can obtain the equations

$$(\partial_{tx} + \partial_{ty} + \partial_{tz} + \partial_{xy} + \partial_{xz} + \partial_{yz})\,\alpha_n = 0 \tag{2}$$

whose solutions will also be harmonic functions.

We note that of all the quaternions, the most suitable one for solving the three-dimensional problems may be the subspace $V = \{xi + yi + zk\}$ whose elements are vectors of a three-dimensional Euclidean space. In this case, however, the solutions of equations in α_2, α_3 and α_4 will not be harmonic functions.

Let us consider a three-dimensional state of stress in which the volume forces are either equal to zero, or are constant. The state is described by three equations of equilibrium, supplemented by boundary conditions /2/. In order to solve them, we shall introduce the following stress function:

$$\begin{gathered}\sigma_x = 2\partial_{tyyzz}\varphi,\quad \sigma_y = 2\partial_{txxzz}\varphi,\quad \sigma_z = 2\partial_{txxyy}\varphi\\ \tau_{xy} = -\partial_{txyzz}\varphi,\quad \tau_{xz} = -\partial_{txyyz}\varphi,\quad \tau_{yz} = -\partial_{txxyz}\varphi\end{gathered} \tag{3}$$

Since the stresses are constant with respect to time, it follows that the stress function φ must depend linearly on time, i.e. $\varphi(t, x, y, z) = kt + \varphi_0(x, y, z)$ where k is a constant and $\varphi_0(x, y, z)$ is a function which depends on the coordinates.

If the volume forces are zero or constant, the equations of compatibility can be expressed, in the case of a three-dimensional stress state, in the form

$$\nabla^2\Theta = 0,\ (1 + \nu)\,\nabla^2\mathbf{E} + (\partial_{xy} + \partial_{xz} + \partial_{yz})\,\Theta \tag{4}$$

where ν is Poisson's ratio, Θ and $\mathbf{E}$ are the sums of the normal and shear stresses, respectively, and ∇^2 is the Laplace operator. Since the stresses are constant with respect to time, we have

$$\partial_{tt}\Theta = \partial_{tt}\mathbf{E} = \partial_{tx}\Theta = \partial_{ty}\Theta = \partial_{tz}\Theta = 0$$

Supplementing Eqs.(4) with given derivatives so as to obtain equations of the type (1) and (2), we have

$$\Box\,\Theta = 0,\ (1 + \nu)\,\Box\mathbf{E} + (\partial_{tx} + \partial_{ty} + \partial_{tz} + \partial_{xy} + \partial_{xz} + \partial_{yz})\,\Theta = 0$$

From this it follows that Θ and $\mathbf{E}$ are harmonic functions. Therefore $\mathbf{E}$ must have conjugate functions Q, R, T. Then $\mathbf{E} + Qi + Rj + Tk$ will be an analytic function of η and we can therefore write $f(\eta) = \mathbf{E} + Qi + Rj + Tk$. The pentuple integral in η of this function represents another analytic function, say $4\Psi(\eta)$. Denoting the real and imaginary parts of

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$\Psi(\eta)$ by e, q, r and t, we obtain

$$\Psi(\eta) = e + qi + rj + tk = \frac{1}{4}\int\!\!\int\!\!\int\!\!\int\!\!\int f(\eta)\, d^5\eta$$

whence $\Psi^{(5)}(\eta) = \frac{1}{4} f(\eta)$. Moreover,

$$\partial_{ttxyz}e + \partial_{ttxyz}qi + \partial_{ttxyz}rj + \partial_{ttxyz}tk = \Psi^{(5)}(\eta)\, k\, \partial_z \eta = -{}^1/_4 f(\eta)$$

Equating all real parts in the first, second, etc. term, we find that $\partial_{ttxyz}\, e = -{}^1/_4\mathbf{E}$. Since e, q, r and t are conjugate functions, their differentiation with respect to ∂_{txyz} will yield

$$\begin{gathered}
\partial_{txxyz}q = -{}^1/_4\mathbf{E}, \quad \partial_{txyyz}r = -{}^1/_4\mathbf{E}, \quad \partial_{txyzz}t = -{}^1/_4\mathbf{E} \\
\partial_{ttxyz}q = -{}^1/_4 Q, \quad \partial_{txxyz}e = {}^1/_4 Q, \quad \partial_{txyyz}t = {}^1/_4 Q \\
\partial_{txyzz}r = -{}^1/_4 Q, \quad \partial_{ttxyz}r = -{}^1/_4 R, \quad \partial_{txxyz}t = -{}^1/_4 R \\
\partial_{txyyz}e = {}^1/_4 R, \quad \partial_{txyzz}q = -{}^1/_4 R, \quad \partial_{ttxyz}t = -{}^1/_4\mathbf{T} \\
\partial_{txxyz}r = {}^1/_4\mathbf{T}, \quad \partial_{txyyz}q = -{}^1/_4\mathbf{T}, \quad \partial_{txyzz}e = {}^1/_4\mathbf{T}
\end{gathered} \tag{5}$$

Taking into account the fact that φ is a linear function of time, $\partial_{ttxyz}\varphi = 0$ and relations (3) for shear stresses, we can write

$$\mathbf{E} = -(\partial_{ttxyz} + \partial_{txxyz} + \partial_{txyyz} + \partial_{txyzz})\,\varphi = \tau_{xy} + \tau_{xz} + \tau_{yz}$$

From (5) it follows that $\varphi + e + q + r + t$ is a harmonic function. Thus we have, for any stress function φ,

$$\varphi = e_1 - e - q - r - t \tag{6}$$

where e_1 is a harmonic function.

We can similarly show that

$$\begin{gathered}
\Theta = (\partial_{ttxyy} + \partial_{ttxxy} + \partial_{ttxzz} + \partial_{ttxxz} + \partial_{ttyzz} + \partial_{ttyyz} + 2\partial_{txxyy} + \\
2\partial_{txxzz} + 2\partial_{tyyzz})\,\varphi = \sigma_x + \sigma_y + \sigma_z
\end{gathered}$$

Let us express the harmonic function Θ in the form of a sum of six harmonic functions

$$\begin{gathered}
\Theta_1 = (\partial_{ttxyy} + \partial_{txxyy})\,\varphi, \quad \Theta_2 = (\partial_{ttxxy} + \partial_{txxyy})\,\varphi \\
\Theta_3 = (\partial_{ttxzz} + \partial_{txxzz})\,\varphi, \quad \Theta_4 = (\partial_{ttxxz} + \partial_{txxzz})\,\varphi \\
\Theta_5 = (\partial_{ttyzz} + \partial_{tyyzz})\,\varphi, \quad \Theta_6 = (\partial_{ttyyz} + \partial_{tyyzz})\,\varphi
\end{gathered} \tag{7}$$

containing conjugate functions B_n, C_n and D_n. From this we find that

$$f_n(\eta) = \Theta_n + B_n i + C_n j + D_n k \;\; (n = 1, \ldots, 6)$$

are analytic functions of η.

Using arguments analogous to those used in deriving (5), we obtain

$$\varphi = \theta + b + c + d + \theta_0 \tag{8}$$

where θ, b, c and d are conjugate functions and θ_0 is a harmonic function.

We can combine Eqs.(6) and (8) by writing

$$2\mathbf{p}_1 = \theta_1 + \theta_0,\; 2\mathbf{p}_0 = \theta - e,\; 2q_0 = b - q,\; 2r_0 = c - r,\; 2t_0 = d - t$$

We have

$$\varphi = p_1 + p_0 + q_0 + r_0 + t_0 \tag{9}$$

where p_1 is a harmonic function, and p_0, q_0, r_0 and t_0 are suitably chosen corresponding conjugate functions.

Let us introduce in (9) the functions q_1, r_1 and t_1, which are harmonic and conjugate with p, and write

$$\delta(\eta) = p_1 + q_1 i + r_1 j + t_1 k$$

Then we can confirm that the real part of the function

$$(p_0 + q_0 i + r_0 j + t_0 k)(1 - i - j - k) + \delta(\eta)$$

is equal to the right-hand side of Eq.(9). Indeed, we can write the stress function in the form

$$\varphi = \mathrm{Re}\,[\varepsilon(\eta)(1 - i - j - k) + \delta(\eta)] \tag{10}$$

We can also simplify, in turn, Eq.(10) and write it in the form

$$\varphi = \mathrm{Re}\,[\Delta(\eta)],\quad \Delta\eta = p_2 + q_2 i + r_2 j + t_2 k,\quad p_2 = p_0 + q_0 + r_0 + t_0 + p_1$$
$$q_2 = -p_0 + q_0 - r_0 + t_0 + q_1,\quad r_2 = -p_0 + q_0 + r_0 - t_0 + r_1,\quad t_2 = -p_0 - q_0 + r_0 + t_0 + t_1$$

where $\Delta\eta$ is an analytic function, p_2 is a harmonic function and q_2, r_2 and t_2 are appropriately chosen conjugate functions.

Since Eq.(10) expresses φ in terms of $\Delta(\eta)$, it follows that this "hypercomplex potential" can also be used to express the stresses.

The hypercomplex function $f(\eta)$ has the corresponding conjugate function $\bar{f}(\bar{\eta})$. It is clear that $f(\eta) + \bar{f}(\bar{\eta}) = 2\,\mathrm{Re}\,f(\eta)$. Then we have $2\varphi = \Delta(\eta) + \bar{\Delta}(\bar{\eta})$, and

$$\sigma_x = \sigma_y = \sigma_z = 2G,\quad \tau_{xy} = -Gk$$
$$\tau_{xz} = -Gj,\quad \tau_{yz} = -Gi,\quad G = \mathrm{Re}\,\Delta^{(6)}(\eta)$$

Let a region V be given in the rectangular xyz coordinate system, and let a surface σ bounded by a three-dimensional curve λ be given in the region V. Let a vector $F = F_x u + F_y v + F_z w$ be defined at every point of the surface, where F is the resultant force with which the material lying to the left of the surface element $d\sigma$ acts on the material lying to the right of $d\sigma$. The components of the resultant force on the surface σ are:

$$F_x = \iint_\sigma \bar{X}\,d\sigma,\quad F_y = \iint_\sigma \bar{Y}\,d\sigma,\quad F_z = \iint_\sigma \bar{Z}\,d\sigma$$

where $\bar{X}, \bar{Y}$ and $\bar{Z}$ are the components of surface forces per unit area at the given point of the boundary. Substituting into these equations instead of the stresses their values in terms of the stress function and putting $\mathbf{X} = 0$, $Y = -\partial_{yyz}\varphi_0$, $Z = \partial_{yzz}\varphi_0$, we obtain

$$F_x = \iint_\sigma [(\partial_y Z - \partial_z Y)\cos(n, x) + (\partial_z \mathbf{X} - \partial_x Z)\cos(n, y) + (\partial_y Y - \partial_y \mathbf{X})\cos(n, z)]\,d\sigma$$

The surface integral over σ is equal to the curvilinear integral over the boundary λ (Stokes's formula). Taking into account the values of X, Y and Z, we obtain

$$F_x = \int_\lambda -\partial_y(\partial_{yz}\varphi_0)\,dy + \partial_z(\partial_{yz}\varphi_0)\,dz$$

It should be noted that since $\mathbf{X} = 0$, it follows that the three-dimensional contour λ coincides with the contour L_{yz} which represents the projection of the contour λ onto the yz plane. On moving along the contour L_{yz} from the point A_{yz} to B_{yz}, the coordinate dz increases and dy decreases. Hence, dy must be taken with a minus sign and

$$F_x = \int_{(\mathrm{A}_{yz})}^{(\mathrm{B}_{yz})} d(\partial_{yz}\varphi_0) = [\mathrm{Re}\,\Delta^{(3)}(\eta)]_{(\mathrm{A}_{yz})}^{(\mathrm{B}_{yz})}\, i$$

We obtain the analogous formulas for F_y and F_z in the same manner.

The surface σ can be a part of a closed boundary surface which will yield, on intersections by planes, closed boundaries λ in space. Projecting λ onto the xy, xz and yz planes will yield closed boundaries L_{xy}, L_{xz} and L_{yz} in the corresponding planes. Then we shall find that when we move from point $\mathbf{A}$ to point $\mathbf{B}$ of the boundary λ in such a manner that the material along the corresponding coordinate axes remains on the left-hand side at all times, the corresponding forces will be F_x, F_y and F_z. Let us find these forces as functions of L_{xy}, L_{xz} and L_{yz}, in the form

$$F_x u + F_y v + F_z w = \frac{1}{2}\{[f_{11}(L_{xy})u + f_{12}(L_{xy})v] + [f_{21}(L_{xz})u + f_{22}(L_{xz})w] + [f_{31}(L_{yz})v + f_{32}(L_{yz})w]\} \tag{11}$$

where all the coefficients of u, v and w are real functions. Denoting the coordinates of the moving point $\mathbf{B}$ by η_{xy}, η_{xz} and η_{yz} when it is projected onto the corresponding coordinate planes xy, xz and yz, we can write the boundary conditions at the edge of the three-dimensional plane in the form

$$-\mathrm{Re}\,\Delta_{xy}^{(3)}(\eta_{xy}) - \mathrm{Re}\,\Delta_{xz}^{(3)}(\eta_{xz}) - \mathrm{Re}\,\Delta_{yz}^{(3)}(\eta_{yz}) = \Delta \tag{12}$$

where Δ is the right-hand side of Eq.(11). In order to obtain from this equation the three hypercomplex potentials, we replace the hypercomplex variables η_{xy}, η_{xz} and η_{yz}, for any point in the physical region, by new hypercomplex variables ζ_{xy}, ζ_{xz} and ζ_{yz} connected by the relation $\eta_i = \omega_i(\zeta_i)$, $i \in \{xy, xz, yz\}$ where $\omega_i(\zeta_i)$ are appropriately chosen functions of ζ_i. The functions are chosen so that the points $\mathbf{P}_i'$ determined by the corresponding hypercomplex coordinates ζ_i in the planes ζ_i correspond to the points $\mathbf{P}_i$ (or map onto these points) in the corresponding η_i planes. The functions which map conformly will be chosen in such a way that the unit circles $\rho_i = 1$ in the planes ζ_i will map onto the curves L_i. It is also convenient to use here polar coordinates ρ_i, Θ_i instead of rectangular coordinates ξ_i, η_i. The functions must be analytic at every point $\mathbf{P}_i'$ which maps onto the corresponding "material" point $\mathbf{P}_i$. A Laurent expansion over the corresponding coordinate planes can be used to provide such points. Then, any functions of η_i will also be functions of ζ_i, obtained by replacing η_i by $\omega_i(\zeta_i)$, and we have

$$\Delta_i^{(3)}(\eta_i) = \Delta_i^{(3)}[\omega_i(\zeta_i)] = \varphi_i(\zeta_i)$$

Let us express the hypercomplex variables η_i in polar coordinates $\eta_i = r_i\sigma_i$, $\sigma_{xy} = je^{k\Theta_{xy}}$, $\sigma_{xz} = ke^{j\Theta_{xz}}$, $\sigma_{yz} = je^{i\Theta_{yz}}$. We note that σ_i is in fact ζ_i for the characteristic points on unit circles. Thus the right-hand side of Eq.(12) can be expressed as a function of σ_i, and we can write

$$\Delta = jf_{xy}(\sigma_{xy}) + kf_{xz}(\sigma_{xz}) + jf_{yz}(\sigma_{yz})$$

The function $jf_{xy}(\sigma_{xy})$ corresponds to a load applied between the points $\mathbf{A}_{xy}$ and $\mathbf{B}_{xy}$ of the xy plane, which is equal to $F_z k$. Similarly, the functions $kf_{xz}(\sigma_{xz})$ and $jf_{yz}(\sigma_{yz})$ correspond to the loads $F_y j$ and $F_x i$.

After all this the boundary condition (12) will take the following form:

$$-\operatorname{Re}\varphi_{xy}(\sigma_{xy}) - \operatorname{Re}\varphi_{xz}(\sigma_{xz}) - \operatorname{Re}\varphi_{yz}(\sigma_{yz}) = jf_{xy}(\sigma_{xy}) + kf_{xz}(\sigma_{xz}) + jf_{yz}(\sigma_{yz}) \tag{13}$$

The three-dimensional boundary condition given is an analogue of a two-dimensional condition in curvilinear coordinates /3/. We shall carry out an additional discussion for this solution.

Let us consider one of the hypercomplex planes, for example the xy plane. Let L be a rectilinear curve in xy, and $\eta_{xy} = \lambda(t) = x(t)\,i + y(t)\,j$ its equation in which t varies from α and β. We shall choose an arbitrary monotonic sequence of $n+1$ values of t: $t_0 = \alpha$, t_1, $\ldots$, $t_n = \beta$. A decomposition $\mathbf{T}$ of the curve L into n arcs $l_0, \ldots, l_{(n-1)}$ corresponds to this choice. Let $f(\eta_{xy}) = u(x, y)\,i + v(x, y)\,j$ be a function, single-valued and continuous on L. We shall choose a single value of the parameter $t = \tau_k$; between t_k and t_{k+1}, and obtain a single point $\zeta_k = \lambda(\tau_k) = \xi_k i + \eta_k j$ on every arc l_k. Then we can show that

$$\lim_{\delta_{\mathbf{T}} \to 0} \sum_{k=0}^{n-1} f(\zeta_k)(\eta_{k+1} - \eta_k) = \int_L f(\eta_{xy})\,d\eta_{xy}$$

Let L be a smooth curve. This will mean that we can find for it a parametric representation $\eta_{xy} = \lambda(t) = x(t)\,i + y(t)\,j$ $(\alpha \leqslant t \leqslant \beta)$, such, that $\lambda(t)$ will have a derivative continuous and not vanishing on the segment $[\alpha, \beta]$. Then the following formula will hold:

$$\begin{gathered}
\int_L f(\eta_{xy})\,d\eta_{xy} = \int_L -u\,dx - v\,dy + k\int_L -v\,dx + u\,dy = \\
\int_\alpha^\beta \{-u[x(t), y(t)]\,x'(t) - v[x(t), y(t)]\}\,dt + \\
k\int_\alpha^\beta \{-v[x(t), y(t)] + u[x(t), y(t)]\}\,dt = \int_\alpha^\beta f[\lambda(t)]\,\lambda'(t)\,dt \\
f[\lambda(t)] = iu[x(t),\ y(t)] + jv[x(t),\ y(t)], \\
\lambda'(t) = ix'(t) + jy'(t)
\end{gathered} \tag{14}$$

where we have utilized the following property of the integrals:

$$\int_L \sum_{k=1}^{p} c_k f_k(\eta_{xy})\,d\eta_{xy} = \sum_{k=1}^{p} c_k \int_L f_k(\eta_{xy})\,d\eta_{xy}$$

It can also be shown that the integral theorem and integral Cauchy formula (14) for the hyperplanes i will also hold.

If we assume that $\eta_i = \omega_i(\zeta_i)$ for $\omega_i(\zeta_i) = R_i(\zeta_i) + m_i/\zeta_i$, where R_i are any positive constants and m_i are positive constants less than unity, the coordinates in the corresponding planes will be expressed in terms of relations known for the ellipses.

Let us determine the potentials $\varphi_i(\zeta_i)$ which satisfy the boundary conditions (13) for any point ζ_i outside the unit circle. We multiply Eq.(13) term by term by $1/(\sigma_i - \zeta_i)$. After this every term will remain a function of σ_i and can therefore be integrated over the unit circles γ_i. Using the Cauchy integral theorem for unit circles, when ζ_i denote the points outside them, we obtain

$$\operatorname{Re}\varphi_i(\zeta_i) = \frac{b_i}{4\pi}\int_{\gamma_i} \frac{f_i(\sigma_i)}{\sigma_i - \zeta_i}\, d\sigma_i, \quad b_{xy} = \frac{j}{k}, \quad b_{xz} = \frac{k}{j}, \quad b_{yz} = \frac{j}{i}$$

Let us consider a space with an elliptical cavity which is stress-free and where the stresses are caused by the application of uniform tensile force S whose projections on the coordinate axes are S_x, S_y and S_z.

We shall seek the analytic potentials, which should be applied to the field of simple tension, acting everywhere in the region without a hole.

Let us consider the xy plane. The force transmitted through the arc $\mathbf{A}_{xy}\mathbf{B}_{xy}$ in accordance with what was said before,

$$F_z^\circ k = S_z\eta_{xy} = S_z R_{xy}\left(\sigma_{xy} + \frac{m_{xy}}{\sigma_{xy}}\right)$$

We further have

$$jf_{xy}(\sigma_{xy}) = F_z k = -F^c k = -S_z R_{xy}\left(\sigma_{xy} + \frac{m_{xy}}{\sigma_{xy}}\right)$$

Substituting this expression into (15) and taking the integral Cauchy theorem into account, we obtain

$$\operatorname{Re}\varphi_{xy}(\zeta_{xy}) = -\frac{1}{2} S_z R_{xy} m_{xy}/\zeta_{xy}$$

and since $F_z k = [\operatorname{Re}\Delta^{(3)}(\eta)]_{(\mathbf{A}_{xy})}^{(\mathbf{B}_{xy})} k = [\operatorname{Re}\Delta^{(3)}(\eta_{xy})]\, k = [\operatorname{Re}\varphi_{xy}(\zeta_{xy})]\, k$, we have

$$\sigma_z = +\frac{2S_z m_{xy}\zeta_{xy}^{3}}{R_{xy}(\zeta_{xy}^2 - m_{xy})^3}, \quad \tau_{xy} = -\frac{S_z m_{xy}\zeta_{xy}^3}{R_{xy}(\zeta_{xy}^2 - m_{xy})^3}\, k$$

The remaining normal and shear stresses are obtained in the same manner when considering the xz and yz planes.

REFERENCES

1. VAN DER WARDEN B.L., Algebra. Nauka, Moscow, 1976.
2. TIMOSHENKO S.P. and GOODIER G., Theory of Elasticity Nauka, Moscow, 1979.
3. MUSKHELISHVILI N.I., Certain Fundamental Problems of the Mathematical Theory of Elasticity. Izd-vo, Akad. Nauk, SSSR, Moscow, 1954.

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